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A Survey of Inductive Systems

BY THEO A. F. KUIPERS

1. INTRODUCTION

In *Studies in Inductive Probability* . . .[1] we reformulated Carnap's program of inductive logic (or probability) in terms of "rational expectation with respect to theories and (outcomes of) experiments". It turned out to be convenient to formulate the "pure systems" behind Carnap's λ-continuum[2] and Hintikka's α-λ-continuum[3] in set theoretical terms. In other words, the "logico-linguistic" approaches of Carnap are set aside, and the assignment of probabilities to the sentences of a language is reconstructed as a particular type of application of pure systems.

Besides two chapters on pure systems, the main chapters in the book are concerned with a general characterization of rational expectation patterns and with a precise characterization of the contexts in which two main types of pure systems (C-systems, corresponding to the λ-continuum, and (open) NH-systems, described by Hintikka and Niiniluoto in article 7 above) give rise to such rational patterns of expectation.

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[1] Theo A. F. Kuipers, *Studies in Inductive Probability and Rational Expectation*, Synthese Library 123 (Dordrecht: D. Reidel, 1978).

[2] Rudolf Carnap, *The Continuum of Inductive Methods* (Chicago: University of Chicago Press, 1952).

[3] Jaakka Hintikka, "A two-dimensional continuum of inductive methods", in *Aspects of Inductive Logic*, ed. J. Hintikka and P. Suppes (Amsterdam: North-Holland, 1966), pp. 113–132.

In this article we shall formulate, without proofs, only the main results of the chapters on pure systems as far as the axiomatic relations and the "inductive" properties are concerned. Because of the reconstructive character, a number of results are reformulations of known results. Definitions and terminology differ slightly from that in the book, in order to avoid complications in this article.

The proof of a main theorem (T9 below) is also given in "On the generalization of the continuum of inductive methods".[4] That theorem states that the *NH*-systems are in fact systems belonging to the class of background systems (*H*-systems) of the α-λ-continuum.

2. CONSISTENT PROBABILITY PATTERNS

Let K be a nonempty finite set $\{Q_1, Q_2, \ldots, Q_k\}$. Let e_n indicate an arbitrary element and E_n an arbitrary subset of K^n, i.e., the nth Cartesian product of K. By $n_i(e_n)$ we indicate the number of occurrences of Q_i in e_n. As variables for *nonempty* subsets of K we use W and V. The size of a subset of K is indicated by the corresponding small letter, e.g., $|W| = w$.

In addition, we use the following abbreviations: $M(e_n) = \{Q_i : n_i(e_n) > 0\}$; $H_W(n) = \{e_n : M(e_n) = W\}$ with $n \geqq w$; finally, $H_W = \bigcup_{n=w}^{\infty} H_W(n)WWW\ldots$ Note that H_W contains those infinite sequences in which all members of W occur at least once and members outside W never occur. Because there is a 1–1-correspondence between these subsets of K^∞ and the so-called constituents in the application-context of "A two dimensional continuum",[5] and "An axiomatic foundation",[6] we call the former *constituents* also. The constituents H_W constitute a (finite) partition of K^∞, i.e., they are mutually nonoverlapping and together exhaustive.

If misunderstandings are excluded, we write simply n_i and M, instead of $n_i(e_n)$ and $M(e_n)$.

Def. 1. A *consistent probability pattern* (CPP) w.r.t. $K, K^2, K^3, \ldots$ is a real-valued function for which:

(1.1) $$p(e_n) \geqq 0$$

[4] Theo A. F. Kuipers, "On the generalization of the continuum of inductive methods to universal hypotheses", in *Synthese* 37.3 (1978), pp. 255–284.

[5] Jaakko Hintikka, "A two-dimensional continuum".

[6] Jaakko Hintikka and Ilkka Niiniluoto, "An axiomatic foundation".

(1.2) $$\sum_{e_n \in K^n} p(e_n) = 1$$

(2) $$\sum_{Q_i \in K} p(e_n Q_i) = p(e_n)$$

(3) $$p(E_n) = \sum_{e_n \in E_n} p(e_n)$$

The pattern is *regular* (RCPP) if (1.1) may be replaced by (1.1R) $p(e_n) > 0$.

A *special value* of a CPP is defined by

(4) $$p(Q_i/e_n) = p(e_n Q_i)/p(e_n) \qquad (p(e_n) \neq 0)$$

Similarly we use e.g., the abbreviations: $p(Q_iQ_j/e_n) = p(e_nQ_iQ_j)/p(e_n)$ and $p(MM/e_n) = p(e_nMM)/p(e_n)$. However, $p(Q_i/e_nM)$, $Q_i \in M$, is short for $p(e_nQ_i)/p(e_nM)$. All probability expressions with '$./e_n$' are supposed to include e_0 in a straightforward way, e.g., $p(Q_i/e_0) = p(Q_i)$.

It is easy to see that a CPP is completely determined by its special values (if $p(e_n) = 0$ then also $p(e_nQ_i) = 0$) and they satisfy

(5.1) $$p(Q_i/e_n) \geqq 0$$

(5.2) $$\sum_{Q_i \in K} p(Q_i/e_n) = 1$$

for all e_n for which $p(e_n) \neq 0$. In a RCCP (5) holds unconditionally and the equality-sign in (5.1) may be omitted.

The well-known extension theorem of Kolmogorov guarantees that the definition, on the basis of a CPP,

(6) $$p(E_nKKK\ldots) = p(E_n)$$

leads to a unique probability measure on the set of (measurable) subsets of K^∞, e.g., $p(H_W)$ is uniquely determined by (6). Because the constituents constitute a partition we have

(7.1) $$p(H_W) \geqq 0$$

(7.2) $$\sum_{W \subset K} p(H_W) = 1$$

The function $p(H_W)$ is called *the prior constitutional distribution* of the CPP. If $p(H_K)=1$ (and hence $p(H_W)=0$ for all $W \neq K$) the CPP is said to be *closed*, otherwise it is called *open*.

If $p(H_W) \neq 0$, we define for $e_n \in W^n$ and $Q_i \in W$

(8) $$p_W(e_n)=p(H_W \cap e_n WWW\ldots)/p(H_W)$$

(9) $$p_W(Q_i/e_n)=p_W(e_nQ_i)/p_W(e_n)$$

It is easy to check that p_W is a CPP w.r.t. W, W^2, $W^3, \ldots$ and that it is closed: $p_W(H_W)=1$. Moreover we have

(10) $$p(e_n)=\sum_{W \supset M} p(H_W)p_W(e_n)$$

From (10) it follows that in case of an RCCP we may add to (7):

(7.3) $$p(H_K)>0$$

We may conclude from this analysis that a CCP can also be determined by the prior constitutional distribution and (the special values of) the relativized patterns.

On the bases of the *relativized patterns* p_W we define *the posterior constitutional distribution*, of course, as

(11) $$p(H_W/e_n)=p(H_W)p_W(e_n)/p(e_n) \qquad W \supset M$$

which is easily seen to be a probability distribution on the H_W with $W \supset M$.

3. C-SYSTEMS

Let $0(e_n, e'_n)$ indicate that e'_n is a permutation of (the order of) e_n. Hence, 0 is an equivalence-relation in K^n, for all n. Consider the principles

SPOI *Strong principle of order indifference*

$$p(e'_n)=p(e_n) \quad \text{if} \quad 0(e_n, e'_n)$$

POI $$p(Q_iQ_j/e_n)=p(Q_jQ_i/e_n)$$

POI′ $$p(Q_i/e'_n)=p(Q_i/e_n) \quad \text{if} \quad 0(e_n, e'_n)$$

These principles are related by the following theorem:

T1. A CPP satisfies SPOI iff it satisfies POI and POI′ (the proof of the if-clause is not trivial).

All systems to be presented satisfy SPOI. Consider now the principle

PERR *Principle of equal restricted relevance*

$$p(Q_i/e_n) = f(n, n_i)$$

(i.e., $p(Q_i/e_n)$ depends only on n and n_i.) Note that PERR implies POI′. The axiomatic introduction of the λ-continuum is based on

T2. If $k > 2$, a CPP satisfying POI and PERR has special values according to

(12) $$p(Q_i/e_n) = (n_i + \lambda/k)/(n + \lambda)$$

in which λ is a real number with $0 \leqq \lambda \leqq \infty$.

Def. 2. A *Carnapian system* (C-system) is a RCPP for which there is a real number λ, $0 < \lambda < \infty$, such that (12) holds. Note that the exclusion of $\lambda = 0$ guarantees that C-systems are regular. The next theorem is of fundamental importance:

T3. A C-system is closed, i.e., $p(H_K) = 1$ (and $p(H_W) = 0$, $W \neq K$). It has frequently been said that C-systems assign zero probability to all nontrivial generalizations. In our approach it becomes clear that this statement is highly problematic, for $p(H_K) = 1$ implies the expectation that *all* members of K will occur sooner or later. Note that a C-system "remains closed": $p(H_K/e_n) = 1$ for all e_n.

The main confirmation properties of C-systems are

instantial confirmation

(13) $p(Q_i/e_nQ_i) > p(Q_i/e_n)$

strong universal-instance confirmation

(14) $p(M/e_nQ_i) > p(M/e_n) \qquad Q_i \in M$

The last property implies

universal-instance confirmation

(15) $p(M/e_nM) > p(M/e_n)$

The following convergence properties can be formulated in a precise way

instantial convergence

(16) $p(Q_i/e_n) \rightarrow n_i/n \qquad \text{if } n \rightarrow \infty$

universal-instance convergence

(17) $p(M/e_n) \rightarrow 1 \qquad \text{if } n \rightarrow \infty$ and M remains constant

In this article we call confirmation and convergence properties together: *inductive properties.*

4. *NH*-SYSTEMS

Consider now the following weak variant of PERR (again implying POI′):

WPERR $p(Q_i/e_n) = f_m(n, n_i)$

This principle is equivalent to the conjunction of

NH1. $p(M/e_n) = g(n, m) = 1 - h(n, m) = 1 - p(\bar{M}/e_n)$

NH2.1. $p(Q_i/e_n\bar{M}) = 1/(k-m) \qquad Q_i \notin M$

NH2.2. $p(Q_i/e_nM) = k_m(n, n_i) \qquad Q_i \in M$

The proof of the following theorem is analogous to that of T2

T4. In a RCCP, satisfying POI and WPERR, there is a real number ρ, $-1 < \rho \leqq \infty$, such that

(18) $p(Q_i/e_nM) = (n_i + \rho)/(n + m\rho) \qquad Q_i \in M$

Def. 3. A *Niiniluoto–Hintikka-system* (*NH*-system) is a RCCP, satisfying NH1, NH2.1 and POI, for which there is a real number ρ, $0<\rho<\infty$, such that (18) holds.

T5. A *NH*-system is completely determined by the k parameters ρ, $h(m, m)$, $m=1, 2, \ldots, k-1$.

T6. A *NH*-system with $h(m, m)=(m+\rho)/(m+k\rho)$, $1\leqq m\leqq k-1$, is a *C*-system with $\lambda=k\rho$, and therefore it is also closed.

5. *H*-SYSTEMS, H_s-SYSTEMS, UNIVERSAL SYSTEMS

In Section 2 we saw that a CCP can be obtained by a prior constitutional distribution and closed relativized patterns.

Def. 4.1. A *Hintakka-system* (*H*-system) is a RCPP with *C*-systems as relativized patterns (with λ_W as parameter) such that we can write: $\lambda_W=\lambda_w$ (and $\rho_W=_{Df}\lambda_W/w=\rho_w$) and with prior constitutional distribution satisfying $p(H_K)>0$ and

H3. $$p(H_W)=q(w)$$

The principles leading to the relativized patterns are:

H1. $$p_W(Q_i/e_n)=f^w(n, n_i) \qquad e_nQ_i\in W^{n+1}$$

H2. $$p_W(Q_iQ_j/e_n)=p_W(Q_jQ_i/e_n) \qquad e_nQ_iQ_j\in W^{n+2}$$

which correspond to PERR and POI resp.

T7. A *H*-system satisfies SPOI.

Def. 4.2. A *special H-system* (H_s-system) is a *H*-system with constant $\rho_W(=\rho)$.

A H_s-system can be based on H1, H2, H3 and

H4. $$p_W(Q_i/e_nM)=p_V(Q_i/e_nM) \qquad Q_i\in M,\ W\supset M,\ V\supset M$$

T8. A *H*-system is determined by $2k-2$ parameters: $q(w)$, λ_w (or ρ_w), $1\leqq w\leqq k-1$; a H_s-system by k-parameters: $q(w)$, $1\leqq w\leqq k-1$ and ρ.

The next theorem, of which the proof is rather complicated, simplifies the study of NH-systems very much

T9. NH-systems are H_s-systems, and vice versa.

In relation with C-systems we have

T10. **1.** A closed H-system is, of course, a C-system
2. An open H_s-system satisfies

$$\textbf{(19)} \qquad p(M/e_n) > (m+\rho)/(m+k\rho) \qquad M \neq K$$

The values at the right side are called the corresponding C-values. It is important to remark that the $h(m, m)$-parameters of T5 have to be in accordance with (19) to guarantee an open NH-system, but that this condition is not sufficient. The necessary and sufficient condition is rather complex and follows from the proof of T9, which will not be given here.

We call a H-system satisfying **UR** $p(H_W) > 0$ for all (nonempty) W a Universal (Regular) system (U-system). We call a H_s-system with this property a special U-system (U_s-system). Note that U-systems are *open* (H-)systems.

U-systems share with C-systems the properties of instantial and universal-instance confirmation and convergence. U_s-systems share in addition the property of *strong* universal-instance confirmation.

Unlike C-systems, U-systems have the following properties:

universal confirmation

$$\textbf{(20)} \quad p(H_M/e_nM) > p(H_M/e_n) \qquad M \neq K$$

universal convergence

$$\textbf{(21)} \quad p(H_M/e_n) \to 1 \qquad \text{if } n \to \infty \text{ and } M(\neq K) \text{ remains constant.}$$

U_s-systems satisfy, in addition,

strong universal confirmation

$$\textbf{(22)} \quad p(H_M/e_nQ_i) > p(H_M/e_n) \qquad M \neq K, Q_i \in M$$

Hintikka's α-λ-continuum is a class of H-systems with a particular prior constitutional distribution which satisfies UR; hence it are U-systems with all listed inductive properties. However, in this article we shall not pay attention to prior distributions, except to one in the next section.

6. STRUCTURAL-INDIFFERENT SYSTEMS

Let $\ddot{e}_n$ indicate the *order-equivalence-class* (with respect to 0) of order-permutations of e_n. Consider the principle

PSI *principle of structural indifference*

$$p(\ddot{e}_n) = p(\ddot{e}'_n)$$

T11. A CPP satisfying SPOI and PSI is a C-system with $\lambda = k$, indicated as C^*-system.

Consider next the weak variant of PSI:

WPSI $p(\ddot{e}_n) = p(\ddot{e}'_n)$ if $m(e'_n) = m(e_n)$

T12. A CPP satisfying SPOI and WPSI is a H_s-system with $\rho = 1$, indicated as a H^*-system (or as U^*-system if UR holds).

If $m(e'_n) = m(e_n)$ we say that $\ddot{e}'_n$ and $\ddot{e}_n$ *have the same size*.

In a discussion of Hintikka's α-λ-continuum, Carnap suggested a very interesting prior constitutional distribution. Let Z_w indicate the union of the sets H_W with $|W|$ equal to a fixed w. Z_w is called a *structure*. In a H-system we have, because of H3,

(23) $$p(Z_w) = \binom{k}{w} q(w)$$

Carnap suggested choosing $q(w)$ so that $p(Z_w) = 1/k$. Let a $^*U^*$-system be a U_s-system with $\rho = 1$ and this particular prior distribution. These systems do only depend on k, i.e., the size of K. In our opinion we may say that, for given K, the (unique) $^*U^*$-system is the most sophisticated application of the so-called (classical) principle of indifferences: equal probabilities for order-permutations, for order-equivalence-classes of the same size and for the structures.

7. GENERALIZED SYSTEMS

Def. 5. A Generalized C-system (GC-system) is a RCCP for which there are real numbers $\gamma_i > 0$, $\sum \gamma_i = 1$, and λ, $0 < \lambda < \infty$, such that

(24) $$p(Q_i/e_n) = (n_i + \gamma_i\lambda)/(n + \lambda)$$

T13. A GC-system satisfies POI and

PRR $p(Q_i/e_n) = f_i(n, n_i)$ (with $f_i(0, 0) = \gamma_i$)

and, consequently, POI′ and SPOI. *Moreover, it is closed.*

Def. 6. **1.** A *Generalized H-system* (GH-system) is a RCCP with GC-systems as relativized patterns (with parameters γ_i^W and $\lambda_W(\rho_W = \lambda_W/w)$) and with an arbitrary prior constitutional distribution with $p(H_K) > 0$.

2. A *special GH-system* (GH_s-system) is a GH-system satisfying, with $\gamma^W =_{\mathrm{Df}} \sum_{Q_i \in W} \gamma_i^K$, $\lambda_W = \lambda_K \cdot \gamma^W$ $\left(\text{or } \rho_W = \frac{k}{w} \rho_K \gamma^W\right)$ and $\gamma_i^W = \gamma_i^K/\gamma^W$.

T14. A GH-system satisfies SPOI.

T15. A closed GH-system is, of course, a GC-system.

The GH_s-systems can directly be seen as generalization of the NH-systems: they arise from POI and

WPRR $$p(Q_i/e_n) = f_{i,M}(n, n_i)$$

Of course, we speak of GU- resp. GU_s-systems if UR is satisfied.

All treated inductive properties of C-, U-, and U_s-systems hold also for GC-, GU-, and GU_s-systems, respectively.

We conclude with a diagram of all treated systems, in which an arrow indicates inclusion.

$$\begin{tikzcd}
GH \arrow[rrr] \arrow[dr] \arrow[dd] & & & GH_s \arrow[rr] \arrow[dr] \arrow[dd] & & GC \arrow[dd] \\
& GU \arrow[rrr] \arrow[dd] & & & GU_s \arrow[dd] & \\
H \arrow[rrr] \arrow[dr] & & & NH = H_s \arrow[rr] \arrow[dr] \arrow[dd] & & C \arrow[dd] \\
& U \arrow[rrr] & & & U_s \arrow[dd] & \\
& & & H^* \arrow[rr] \arrow[dr] & & C^* \\
& & & & U^* \arrow[d] & \\
& & & & {}^*U^* &
\end{tikzcd}$$